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An existence result for a class of second order evolution equations of mixed type

Fabio Paronetto

Dipartimento di Matematica "Ennio De Giorgi", Università degli Studi di Lecce, Via per Arnesano, 73100 Lecce, Italy

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Abstract

We give an existence and uniqueness result for a linear abstract evolution equation of second order with some coefficient in front of the second temporal derivative which may degenerate to zero and change sign.

Keywords: Differential equations of mixed type; Second order evolution equations

1. Introduction

We present an existence result for the solution of the abstract evolution equation of second order

$$(\mathcal{S}u')' + \mathcal{N}u' + \mathcal{A}u = f \tag{1}$$

with suitable data. Given V Banach space and H Hilbert space, $V \subset H$, defined $\mathcal{V} = L^2(0, T; V)$, $\mathcal{S}$ will be a linear operator defined in $L^2(0, T; H)$ and the solution u will be taken in $C([0, T]; V)$, with $u' \in \mathcal{V}$ and $(\mathcal{S}u')' \in \mathcal{V}'$. Notations and definitions for the abstract equation will be given in the second section, so for the moment we confine ourselves to explain a concrete and simple problem (see also Section 3 and [8]): suppose to have a bounded open set Ω of $\mathbf{R}^n$, $T > 0$, a function $s : \Omega \times [0, T] \to \mathbf{R}$, $s \in L^\infty(\Omega \times (0, T))$, define $\Omega_+(t) = \{x \in \Omega \mid s(x, t) > 0\}$ and $\Omega_-(t) = \{x \in \Omega \mid s(x, t) < 0\}$, consider $H = L^2(\Omega)$ and $V = H_0^1(\Omega)$, and define $\mathcal{S}u(x, t) = s(x, t)u(x, t)$. Consider

E-mail address: fabio.paronetto@unile.it.

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$$\begin{cases} (s(x,t)u_t(x,t))_t - \Delta u_t(x,t) - \Delta u(x,t) = f(x,t) \quad \text{on } \Omega \times (0,T), \\ u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \\ u_t(x,0) = \varphi(x), \quad x \in \Omega_+(0), \\ u_t(x,T) = \psi(x), \quad x \in \Omega_-(T), \\ u(x,0) = \eta(x), \quad x \in \Omega, \end{cases} \tag{2}$$

where $f : \Omega \times (0,T) \to \mathbf{R}$, $\varphi : \Omega_+(0) \to \mathbf{R}$, $\psi : \Omega_-(T) \to \mathbf{R}$, $\eta : \Omega \to \mathbf{R}$ are the data of the problem. Notice that for this kind of problem we prescribe an initial datum for the function u in the whole Ω, but for the function u_t only in the part in which at time $t = 0$ the coefficient s is positive, while at time $t = T$ where s is negative. Where $s \equiv 0$ no data for u_t are imposed. These requirements about initial/final data for u_t are quite natural and some comments about them will be done in Section 3.

Equation like that in (2) are widely studied. The classical case corresponds to take $s \equiv 1$ (or s strictly positive) in (2), i.e. the wave equation with dissipation or friction (see, for instance, [16, Section 33.5]).

Also the theory of elasticity (see, for instance, [4, Chapter 3]) leads to an equation like

$$u_{tt} = \operatorname{div}\bigl(a(Du)\bigr),$$

in which often an additional dissipative term is added (a term $\epsilon \Delta u_t$ with ϵ positive and little parameter) to regularise the solution, so that the equation becomes (see, e.g., [10,11,15] and, for a non-monotone, for instance, [9])

$$u_{tt} = \operatorname{div}\bigl(a(Du) + \epsilon D u_t\bigr). \tag{3}$$

Moreover the quasi-steady approximation, which correspond to consider $s \equiv 0$, i.e.

$$\operatorname{div}\bigl(a(Du) + \epsilon D u_t\bigr) = 0, \tag{4}$$

has been also considered for equation of this type (we refer, e.g., for this case, to [12]).

Changing type equations have been already studied a lot. As regards evolution equations of first order (partially elliptic, partially parabolic) we recall some classical books (see [2,14]) where many degenerate problems are presented in which the coefficients in front of temporal derivatives may be non-negative. Among the situations in which the coefficient in front of temporal derivative may be also negative we recall [1,7,8].

As regards the degenerate situation for second order equations like that in (2) we recall [13, see in particular Chapter VI] for $s \geqslant 0$ (one can think there are two regions, one where the evolution is standard like in (3) and a region where the evolution is quasi-steady like in (4)).

A well-known and studied situation of changing type equation of second order is the Tricomi equation

$$x u_{tt} - u_{xx} = 0,$$

or more generally $s(x)u_{tt} - u_{xx} = 0$, which occurs when studying transonic flow, for which we refer to the recent and also historical paper [6] (see also the references therein) and to [3].

In the present paper we analyse the situation considered by Showalter in [13], genaralising (see Theorem 3.4) the existence result in the sense that the coefficient s may be not only non-negative, but positive, null and negative.

To do this, we reduce by a change of variable to a first order evolution equation and use an existence result for mixed equation of first order (see [8]) which follows by a classical result for perturbation of monotone operators (Theorem 2.5). Although in [8] the operator $\mathcal{A}$ in (1) may be non-linear, here we manage to consider $\mathcal{A}$ only linear (this is due to Proposition 2.3, needed when we reduce to a first order equation).

With our method we do not manage to generalise the Tricomi equation (which correspond to take $\mathcal{N} = 0$ in (1)), neither via regularisation, i.e. taking the solutions of $(\mathcal{S}u')' + \epsilon\mathcal{N}u' + \mathcal{A}u = f$ and then let ϵ go to zero (see, for instance, [5, Chapter 3, Section 8.5] for the analogous classical result) due to the change of sign of $\mathcal{S}$.

At the end we comment a concrete example, where the operator $\mathcal{S}$ is defined by a function s like in (2), referring to [8] for examples of possible choices of the function s.

2. Recalls and notations

We want to give an existence result for the abstract equation

$$(\mathcal{S}u')' + \mathcal{N}u' + \mathcal{A}u = f$$

($\mathcal{S}, \mathcal{N}, \mathcal{A}$ are operators) with suitable "initial/final" conditions. In this section we give some recalls, assumptions about these three operators and some results contained in [8].

Consider a triplet

$$V \subset H \subset V'$$

where V is a reflexive and separable Banach space, V' its dual space, H a Hilbert space and the embedding $V \subset H$ is continuous and dense. Fix $T > 0$ and denote

$$\mathcal{V} = L^2(0, T; V), \qquad \mathcal{H} = L^2(0, T; H), \qquad \mathcal{V}' = L^2(0, T; V').$$

Consider a family of linear self-adjoint operators

$$S(t) : H \to H, \quad t \in [0, T].$$

For every $t \in [0, T]$ consider the decomposition

$$H = H_+(t) \oplus H_0(t) \oplus H_-(t)$$

where $H_0(t)$ is the kernel of $S(t)$, $H_+(t)$ and $H_-(t)$ are defined respectively as the subspaces of H such that $(S(t)u, u)_H > 0$ for every $u \in H_+(t)$, $u \neq 0$, and $(S(t)u, u)_H < 0$ for every $u \in H_-(t)$, $u \neq 0$, and decompose $S(t)$ as

$$\begin{aligned}
&S(t) = S_+(t) + S_0(t) - S_-(t) \quad \text{where} \\
&\quad S_+(t) : H_+(t) \to H_+(t) \quad \text{defined by} \quad S_+(t)u = S(t)u \quad \text{for } u \in H_+(t), \\
&\quad S_-(t) : H_-(t) \to H_-(t) \quad \text{defined by} \quad S_-(t)u = -S(t)u \quad \text{for } u \in H_-(t)
\end{aligned} \tag{5}$$

and $S_0(t): H_0(t) \to H_0(t)$ is the operator $S_0(t)u = 0$ for every $u \in H_0(t)$. Moreover we define

$$\left|S(t)\right|: H \to H \quad \text{by} \quad \left|S(t)\right|u = S_+(t)u + S_-(t)u.$$

Now denote

$$\tilde{H}(t), \tilde{H}_+(t), \tilde{H}_-(t) \quad \text{the completion respectively of} \quad H, H_+(t), H_-(t) \tag{6}$$

with respect to the norm $\|w\|_{\tilde{H}(t)} = \||S(t)|^{1/2}w\|_H$ and denote the orthogonal projections

$$P_+(t): \tilde{H}(t) \to \tilde{H}_+(t), \qquad P_-(t): \tilde{H}(t) \to \tilde{H}_-(t), \qquad P_0(t): \tilde{H}(t) \to H_0(t)$$

for every $t \in [0, T]$ and in particular we will need

$$P_+(0): \tilde{H}(0) \to \tilde{H}_+(0) \quad \text{and} \quad P_-(T): \tilde{H}(T) \to \tilde{H}_-(T). \tag{7}$$

Before giving assumptions about S we recall a definition.

Definition 2.1. We say that $B: [0, T] \to \mathcal{L}(X, X')$, X Banach space, is differentiable if, for every $u, v \in X$ the function

$$t \mapsto \langle B(t)u, v\rangle_{X' \times X}$$

is absolutely continuous on $[0, T]$ and there exists a function $b \in L^1(0, T)$ such that

$$\left|\frac{d}{dt}\langle B(t)u, v\rangle_{X' \times X}\right| \leqslant b(t)\|u\|_X\|v\|_X \quad \text{for a.e. } t \in [0, T].$$

Observe that $B'(t): X \to X'$ is linear for almost every $t \in (0, T)$.

Remark 2.2. If B is differentiable in the sense just defined, the following formula holds: if $u \in W^{1,p}(0, T; X)$ for any p, then

$$\big(B(t)u(t)\big)' = B'(t)u(t) + B(t)u'(t) \quad \text{for a.e. } t \in (0, T).$$

In the special case $B(t) = B$ for every t, taking $v \in L^p(0, T; X)$ and $u(t) = \int_0^t v(s)\,ds$, we also derive from the formula above that

$$B\int_0^t v(s)\,ds = \int_0^t Bv(s)\,ds. \tag{8}$$

Assumptions about $\mathcal{S}$

We suppose that the family of operators $S(t)$ will satisfy:

$$\begin{aligned}
&\text{(i)} && S(t) \text{ self-adjoint},\\
&\text{(ii)} && \max_{t\in[0,T]} \|S(t)\|_{\mathcal{L}(H)} \leqslant C_1,\\
&\text{(iii)} && t \mapsto \big(S(t)u, v\big)_H \text{ absolutely continuous on } [0,T],\\
&\text{(iv)} && \Gamma\|u\|_V\|v\|_V \leqslant \frac{d}{dt}\big(S(t)u, v\big)_H \leqslant C_2\|u\|_V\|v\|_V \quad \text{for a.e. } t\in[0,T]
\end{aligned} \tag{9}$$

for every $u, v \in V$, for some non-negative constants C_1, C_2 and $\Gamma \in \mathbf{R}$ (observe that the derivative of $(S(t)u, v)_H$ is estimated by the norms of u and v in the space V).

Then we define an operator $\mathcal{S}$

$$\mathcal{S} : L^2(0,T;H) \to L^2(0,T;H) \quad \text{by} \quad \mathcal{S}u(t) := S(t)u(t) \tag{10}$$

which turns out to be linear and bounded by the constant C_1. Since S satisfies (9) we can define a family of equibounded operators

$$S' : [0,T] \to \mathcal{L}(V, V') \quad \text{by} \quad \big\langle S'(t)u, v\big\rangle_{V'\times V} := \frac{d}{dt}\big(S(t)u, v\big)_H$$

and an operator

$$\mathcal{S}' : \mathcal{V} \to \mathcal{V}' \quad \text{by} \quad \langle \mathcal{S}'u, v\rangle_{\mathcal{V}'\times\mathcal{V}} := \int_0^T \big\langle S'(t)u(t), v(t)\big\rangle_{V'\times V}\, dt$$

which turns out to be linear and bounded by $\max\{|\Gamma|, C_2\}$.

Assumptions about $\mathcal{N}$

The only requirement about the operator $\mathcal{N}$

$$\mathcal{N} : \mathcal{V} \to \mathcal{V}'$$

is boundedness, i.e. there is a constant C_3 such that

$$\|\mathcal{N}u\|_{\mathcal{V}'} \leqslant C_3\|u\|_{\mathcal{V}}. \tag{11}$$

Assumptions about $\mathcal{A}$

Consider a family of operators:

$$\begin{gathered}
A(t) : V \to V' \quad \text{with} \quad t \mapsto \big\langle A(t)u, v\big\rangle_{V'\times V} \text{ measurable on } [0,T],\\
A(t) \text{ linear, monotone and symmetric for a.e. } t\in[0,T],
\end{gathered}$$

such that if we define the abstract operator

$$\mathcal{A}:\mathcal{V}\to\mathcal{V}', \quad \mathcal{A}u(t)=A(t)u(t), \quad 0\leqslant t\leqslant T, \tag{12}$$

this turns out to be linear, monotone and symmetric. We moreover suppose $\mathcal{A}$ differentiable (see Definition 2.1) and denote by $\mathcal{A}'$ the operator

$$\mathcal{A}':\mathcal{V}\to\mathcal{V}' \quad \text{defined by} \quad \mathcal{A}'v(t):=A'(t)v(t).$$

About $\mathcal{A}$ we suppose that there is a non-negative constant C_4, such that

(i) $A(t)$ symmetric for every t,

(ii) $\max_{t\in[0,T]}\|A(t)\|_{\mathcal{L}(V,V')}\leqslant C_4$,

(iii) $\mathcal{A}$ monotone,

(iv) $t\mapsto\langle A(t)u,v\rangle_{V'\times V}$ is absolutely continuous on $[0,T]$,

$$\text{and} \quad \frac{d}{dt}\langle A(t)u,u\rangle_{V'\times V}\leqslant 0 \quad \text{for every } u,v\in V. \tag{13}$$

Consider the operator

$$J:\mathcal{V}\to\mathcal{V}, \quad [Jv](t):=\int_0^t v(\sigma)\,d\sigma, \tag{14}$$

and a family $Q(t):V\to V'$ of linear, monotone and symmetric operators. Denote by $\mathcal{Q}$ the operator $\mathcal{Q}:\mathcal{V}\to\mathcal{V}'$ defined by

$$\mathcal{Q}v(t)=Q(t)v(t).$$

We recall the following proposition.

Proposition 2.3. *Consider $Q(t):V\to V'$ a family of linear, monotone and symmetric operators. Suppose $\mathcal{Q}$ differentiable and denote by $Q'(t):V\to V'$ the operators defined as follows*:

$$\langle Q'(t)w_1,w_2\rangle_{V'\times V}:=\frac{d}{dt}\langle Q(t)w_1,w_2\rangle_{V'\times V}.$$

If $-Q'(t)$ is monotone for a.e. $t\in[0,T]$ then the operator

$$\mathcal{V}\ni v\mapsto\mathcal{Q}Jv(t)=Q(t)\int_0^t v(\sigma)\,d\sigma$$

is monotone. If $\mathcal{Q}$ is bounded, $\mathcal{Q}J$ is bounded by $T\|\mathcal{Q}\|_{\mathcal{L}(\mathcal{V},\mathcal{V}')}$.

Remark 2.4. Analogously one can prove that if $Q'(t)$ is monotone for a.e. $t \in [0,T]$ and we define $J_T v(t) = \int_t^T v(\sigma)\,d\sigma$, then the operator $\mathcal{Q}J : \mathcal{V} \to \mathcal{V}'$ is monotone (if $\mathcal{Q}$ is bounded, also $\mathcal{Q}J$ is bounded).

Proof. The space of polynomials $q(t)$ with coefficients in V and such that $q(0) = 0$ is dense in $\mathcal{V}$. Since $Q(t)$ is linear and symmetric we have

$$\frac{d}{dt}\big\langle Q(t)q(t), q(t)\big\rangle_{V'\times V} = 2\big\langle Q(t)q(t), q'(t)\big\rangle_{V'\times V} + \big\langle Q'(t)q(t), q(t)\big\rangle_{V'\times V}. \tag{15}$$

Integrating (15) in $(0,T)$ if q is a polynomial such that $q(0) = 0$ we have that

$$2\langle \mathcal{Q}q, q'\rangle_{\mathcal{V}'\times\mathcal{V}} = -\langle \mathcal{Q}'q, q\rangle_{\mathcal{V}'\times\mathcal{V}} + \big\langle Q(T)q(T), q(T)\big\rangle_{V'\times V} \geqslant 0$$

and taking $q(t) = Jp(t) = \int_0^t p(\sigma)\,d\sigma$ with p polynomial we conclude.

For the last statement observe that J is continuous and, if $\mathcal{Q}$ is bounded, also $\mathcal{Q}$ is continuous. Then $\mathcal{Q}J$ is linear and continuous. □

Finally we recall the following classical result for which we refer to [16] (see Section 32.4).

Theorem 2.5. *Let $B : X \to X'$ (X' the dual space of X, X Banach space) be continuous, monotone, bounded and coercive, i.e. $\lim_{\|x\|\to+\infty} \|x\|^{-1}\langle Bx, x\rangle \to +\infty$. Suppose $L : X \to X'$ to be maximal monotone. Then for every $f \in X'$ the following equation has a solution*

$$Lu + Bu = f.$$

If moreover B is strictly monotone the solution is unique.

3. The existence result

In the present section we want to study the problem:

$$\begin{cases} (\mathcal{S}u')' + \mathcal{N}u' + \mathcal{A}u = f, \\ P_+(0)u'(0) = \varphi, \\ P_-(T)u'(T) = \psi, \\ u(0) = \eta \end{cases} \tag{16}$$

with $f \in \mathcal{V}'$, $\varphi \in \tilde{H}_+(0)$, $\psi \in \tilde{H}_-(T)$, $\eta \in V$.

The boundary conditions with respect to the variable t, i.e. the initial/final conditions, are given as follows: we give an initial condition for u' at time zero where $\mathcal{S}$ is positive (i.e. the datum φ) while a final condition at time T where $\mathcal{S}$ is negative (i.e. the datum ψ). Where $\mathcal{S}$ is null, no conditions for u' are given.

About u we impose an initial datum (i.e. η) at time zero, but, as we will see, this datum could be given also at time T.

If $\mathcal{S} \equiv 0$ the initial/final conditions make no sense and the problem simply becomes

$$\begin{cases} \mathcal{N}u' + \mathcal{A}u = f, \\ u(0) = \eta. \end{cases} \tag{17}$$

The initial/final conditions we require about u' and u are easily understood by explaining how we prove the existence result: indeed the idea to solve problem (16) is to consider the operator J defined in (14) and the change of variable $v = u'$ in (16) and then solve the first order problem:

$$\begin{cases} (\mathcal{S}v)' + \mathcal{N}v + \mathcal{A}Jv = f - \mathcal{A}\eta, \\ P_+(0)v(0) = \varphi, \\ P_-(T)v(T) = \psi. \end{cases} \tag{18}$$

Then it is natural to impose for the equation, partially elliptic and partially parabolic, both forward and backward, an initial datum where $\mathcal{S}$ is positive, no datum where $\mathcal{S} = 0$, a final datum at time T where $\mathcal{S}$ is negative.

Clearly the initial condition about u is due to the choice of the operator J. In the same way one could fix a condition at time T, that is to consider $u(T) = \eta$, provided we consider the change of variable defined by the operator J_T defined in Remark 2.4.

For our purpose we introduce the space

$$\mathcal{W} = \{v \in \mathcal{V} \mid (\mathcal{S}v)' \in \mathcal{V}'\}, \qquad \|u\|_{\mathcal{W}} = \|u\|_{\mathcal{V}} + \|(\mathcal{S}u)'\|_{\mathcal{V}'} \tag{19}$$

and its subspace (see Proposition 2.6, Remarks 2.7 and 3.1 in [8])

$$\mathcal{W}^0 = \{v \in \mathcal{W} \mid P_+(0)v(0) = 0,\ P_-(T)v(T) = 0\}.$$

Definition 3.1. We say that $v \in \mathcal{W}$ is a solution of problem (18) with $f \in \mathcal{V}'$, $\varphi \in \tilde{H}_+(0)$, $\psi \in \tilde{H}_-(T)$, if

$$(\mathcal{S}v)'(t) + \mathcal{N}v(t) + \mathcal{A}Jv(t) = f(t) \quad \text{in } V' \quad \text{for a.e. } t \in [0,T],$$
$$P_+(0)v(0) = \varphi, \qquad P_-(T)v(T) = \psi.$$

We say that $u \in \{w \in C([0,T]; V) \mid w' \in \mathcal{W}\}$ is a solution of problem (16) with $f \in \mathcal{V}'$, $\varphi \in \tilde{H}_+(0)$, $\psi \in \tilde{H}_-(T)$, $u(0) = \eta$, if

$$(\mathcal{S}u')'(t) + \mathcal{N}u'(t) + \mathcal{A}u(t) = f(t) \quad \text{in } V' \quad \text{for a.e. } t \in [0,T],$$
$$P_+(0)u'(0) = \varphi, \quad P_-(T)u'(T) = \psi, \quad u(0) = \eta.$$

If $\mathcal{S} \equiv 0$ the solution of (17) will be a function in $H^1(0,T;V)$.

Proposition 3.2. *A function $v \in \mathcal{W}$ is a solution of* (18) *if and only if*

$$\begin{aligned} -(\mathcal{S}v, \Phi')_{\mathcal{H}} &+ \langle \mathcal{N}v, \Phi\rangle_{\mathcal{V}'\times\mathcal{V}} + \langle \mathcal{A}Jv, \Phi\rangle_{\mathcal{V}'\times\mathcal{V}} \\ &= \langle f, \Phi\rangle_{\mathcal{V}'\times\mathcal{V}} + \big(S_+(0)\varphi, \Phi(0)\big)_H - \big(S_-(T)\psi, \Phi(T)\big)_H \end{aligned} \tag{20}$$

for every $\Phi \in X = \{v \in \mathcal{V} \mid v' \in \mathcal{H},\ P_-(0)v = 0,\ P_+(T)v = 0\}$.

Proof. Consider the definition given in Definition 3.1 and suppose v is a solution of (18). Multiplying the equation in (18) by a function $\Phi \in X$ we easily obtain (20). Suppose now to have (20) for every $\Phi \in X$. Consider

$$\Phi(t) = \vartheta(t)w + \vartheta_+(t)w_+ + \vartheta_-(t)w_-$$

where $w \in V$, $w_+ \in V \cap (\tilde{H}_+(0) \oplus \tilde{H}_0(0))$, $w_- \in V \cap (\tilde{H}_-(T) \oplus \tilde{H}_0(T))$, $\vartheta(0) = \vartheta(T) = \vartheta_+(T) = \vartheta_-(0) = 0$. Then by (20) we obtain

$$\begin{aligned}
&-\int_0^T \big(\mathcal{S}v(t), \vartheta'(t)w + \vartheta'_+(t)w_+ + \vartheta'_-(t)w_-\big)_H\,dt \\
&\qquad + \int_0^T \big\langle \mathcal{N}v(t) + \mathcal{A}Jv(t), \vartheta(t)w + \vartheta_+(t)w_+ + \vartheta_-(t)w_-\big\rangle_{V'\times V}\,dt \\
&\quad = \int_0^T \big\langle f(t), \vartheta(t)w + \vartheta_+(t)w_+ + \vartheta_-(t)w_-\big\rangle_{V'\times V}\,dt \\
&\qquad + \big(S_+(0)\varphi, \vartheta_+(0)w_+\big) - \big(S_-(T)\psi, \vartheta_-(T)w_-\big).
\end{aligned} \tag{21}$$

Taking $\vartheta_+ = \vartheta_- = 0$ in (21) we obtain

$$\int_0^T \left(\frac{d}{dt}\mathcal{S}v(t), w\right)_H \vartheta(t)\,dt + \int_0^T \big\langle \mathcal{N}v(t) + \mathcal{A}Jv(t), w\big\rangle_{V'\times V}\vartheta(t)\,dt = \int_0^T \big\langle f(t), w\big\rangle_{V'\times V}\vartheta(t)\,dt.$$

Since this is true for every $\vartheta \in C_0^1(0,T)$ (and for every $w \in V$) we obtain that

$$\frac{d}{dt}\big(\mathcal{S}v(t), w\big)_H + \big\langle \mathcal{N}v(t) + \mathcal{A}Jv(t), w\big\rangle_{V'\times V} = \big\langle f(t), w\big\rangle_{V'\times V}$$

for every $w \in V$ and a.e. $t \in [0,T]$. From this equality, (20) and

$$\begin{aligned}
&\big(S(T)v(T), \vartheta_-(T)w_-\big)_H - \big(S(0)v(0), \vartheta_+(0)w_+\big)_H \\
&\quad = \int_0^T \frac{d}{dt}\Big[\big(\mathcal{S}v(t), \vartheta(t)w + \vartheta_+(t)w_+ + \vartheta_-(t)w_-\big)_H\Big]\,dt \\
&\quad = \int_0^T \left[\frac{d}{dt}\big(\mathcal{S}v(t), w\big)_H\vartheta(t) + \frac{d}{dt}\big(\mathcal{S}v(t), w_+\big)_H\vartheta_+(t) + \frac{d}{dt}\big(\mathcal{S}v(t), w_-\big)_H\vartheta_-(t)\right]dt \\
&\qquad + \int_0^T \Big[\big(\mathcal{S}v(t), w\big)_H\vartheta'(t) + \big(\mathcal{S}v(t), w_+\big)_H\vartheta'_+(t) + \big(\mathcal{S}v(t), w_-\big)_H\vartheta'_-(t)\Big]\,dt
\end{aligned}$$

we derive

$$\begin{aligned}&\big(S(T)v(T), \vartheta_-(T)w_-\big)_H - \big(S(0)v(0), \vartheta_+(0)w_+\big)_H\\ &\quad = \big(S_-(T)\psi, \vartheta_-(T)w_-\big)_H - \big(S_+(0)\varphi, \vartheta_+(0)w_+\big)_H\end{aligned}$$

for every ϑ_+, ϑ_-, w_- w_+ as chosen above. Then $P_+(0)v(0) = \varphi$ and $P_-(T)v(T) = \psi$. □

Now observe that if $(\mathcal{S}u)' \in \mathcal{V}'$ and $\mathcal{S}'u \in \mathcal{V}'$, $\mathcal{S}u'$ makes sense in $\mathcal{V}'$. Then, if we introduce the operator

$$\mathcal{L}: D(\mathcal{L}) \subset \mathcal{V} \to \mathcal{V}', \quad \mathcal{L}u = \mathcal{S}u' + \frac{1}{2}\mathcal{S}'u, \quad D(\mathcal{L}) = \mathcal{W}^0,$$

the following result holds (see Proposition 3.2 in [8]).

Proposition 3.3. *The operator $\mathcal{L}: D(\mathcal{L}) \subset \mathcal{V} \to \mathcal{V}'$ is maximal monotone.*

Denote by $\mathcal{B}$ the operator

$$\mathcal{B} = \frac{1}{2}\mathcal{S}' + \mathcal{N} + \mathcal{A}J \tag{22}$$

and by $B(\sigma)$ the operator $B(\sigma) = \frac{1}{2}S'(\sigma) + N(\sigma) + A(\sigma)J(\sigma)$ for $\sigma \in [0, T]$, so that we write the equation $(\mathcal{S}v)' + \mathcal{N}v + \mathcal{A}Jv = f$ in (18) as

$$\mathcal{L}v + \mathcal{B}v = f$$

and use Theorem 2.5 to prove the result just below. At first we define the following spaces:

$$\begin{aligned}V_+(0) &= \big\{\big[P_+(0) + P_0(0)\big]w \in V \mid w \in V\big\} = V \cap \big(\tilde{H}_+(0) \oplus \tilde{H}_0(0)\big),\\ V_-(T) &= \big\{\big[P_-(T) + P_0(T)\big]w \in V \mid w \in V\big\} = V \cap \big(\tilde{H}_-(T) \oplus \tilde{H}_0(T)\big).\end{aligned}$$

Observe that the operator $\mathcal{B}$ is bounded (by (9), (11), (13) and Proposition 2.3) by

$$\|\mathcal{B}u\|_{\mathcal{V}'} \leqslant C_5\|u\|_{\mathcal{V}} \quad \text{for every } u \in \mathcal{V}, \qquad C_5 = \frac{1}{2}C_2 + C_3 + TC_4. \tag{23}$$

Moreover, if we suppose there is a constant $\gamma \in \mathbf{R}$ for which (see (9) for Γ)

$$\langle \mathcal{N}u - \mathcal{N}v, u - v\rangle \geqslant \gamma\|u - v\|^2_{\mathcal{V}} \quad \text{for every } u, v \in \mathcal{V}, \quad \text{and} \quad \gamma + \Gamma/2 > 0, \tag{24}$$

the operator $\mathcal{B}$ satisfies (by Proposition 2.3)

$$\langle \mathcal{B}u - \mathcal{B}v, u - v\rangle \geqslant \left\langle \frac{1}{2}\mathcal{S}'(u - v) + \mathcal{N}u - \mathcal{N}v, u - v\right\rangle \geqslant \left(\frac{\Gamma}{2} + \gamma\right)\|u - v\|^2_{\mathcal{V}}. \tag{25}$$

To prove the following result we will require that

$$H_+(0) \cap H_-(T) = \{0\}. \tag{26}$$

Theorem 3.4. *Suppose $\mathcal{B}$ monotone and coercive. Under assumptions* (9), (11), (13), (26) *for every $f \in \mathcal{V}'$, $\eta \in V$, $\varphi \in V$, $\psi \in V$ problem* (16) *admits one solution $u \in C([0,T];V)$ with $u' \in \mathcal{W}$. If $\mathcal{B}$ is strictly monotone the solution is unique. If moreover*

$$V_+(0) \quad \text{dense in } \tilde{H}_+(0), \qquad V_-(T) \quad \text{dense in } \tilde{H}_-(T), \tag{27}$$

$\mathcal{N}$ is continuous and (25) *holds with $\Gamma/2 + \gamma > 0$ then the data φ and ψ may be chosen respectively in $\tilde{H}_+(0)$ and $\tilde{H}_-(T)$ and the following estimate holds*:

$$\begin{aligned} &\big\|(\mathcal{S}u')'\big\|_{\mathcal{V}'} + \|u'\|_{\mathcal{V}} + \max_{t\in[0,T]}\big\|u(t)\big\|_V \\ &\quad \leqslant \|\eta\|_V + c\big(\|f\|_{\mathcal{V}'} + \big\|S_-^{1/2}(T)\psi\big\|_{H_-(T)} + \big\|S_+^{1/2}(0)\varphi\big\|_{H_+(0)}\big) \end{aligned}$$

where c is a constant depending (*only*) *on $T, C_5, \Gamma/2+\gamma$* (*see* (23) *and* (25)).

Remark 3.5. For details about assumption (27) we refer to Section 4 in [8].

Remark 3.6. One can obtain an analogous result for the problem (16) with the condition $u(T) = \eta$ instead of $u(0) = \eta$ replacing the assumption $\frac{d}{dt}\langle A(t)u, u\rangle_{V'\times V} \leqslant 0$ in (13) with $\frac{d}{dt}\langle A(t)u, u\rangle_{V'\times V} \geqslant 0$ (see Remark 2.4).

In particular if $\mathcal{A}$ is independent of t the problem admits a unique solution both with condition about u at time zero and at time T.

Remark 3.7. Assumption (26) means $\mathrm{supp}(\varphi) \cap \mathrm{supp}(\psi) = \emptyset$ (which is needed in Step 2 of the proof).

Proof. Consider the operator J defined in (14). We can rewrite problem (16) as the first-order evolution problem:

$$\begin{cases} (\mathcal{S}v)' + \mathcal{N}v + \mathcal{A}Jv = f - \mathcal{A}\eta, \\ P_+(0)v(0) = \varphi, \\ P_-(T)v(T) = \psi, \end{cases} \qquad u(t) = \eta + \int_0^t v(\sigma)\,d\sigma \tag{28}$$

in the space $\mathcal{W} = \{v \in \mathcal{V} \mid (\mathcal{S}v)' \in \mathcal{V}'\}$. It is easy to verify that (16) and (28) are equivalent, i.e. if $u \in C([0,T];V)$ and $u' \in \mathcal{W}$ is the solution of (16) then $v = u' \in \mathcal{W}$ is the solution of (28) and, vice versa, if $v \in \mathcal{W}$ is the solution of (28) then $u = \eta + Jv \in C([0,T;V)$ is the solution of (16) and $u' \in \mathcal{W}$ (see, e.g., [16, Section 32.10]).

First observe that the operator $\mathcal{N}$ is strictly monotone, bounded and coercive by (11), $\mathcal{A}J$ is continuous, monotone, bounded and coercive by (13) and Proposition 2.3. Note then that requiring

$$\mathcal{B} = \frac{1}{2}\mathcal{S}' + \mathcal{N} + \mathcal{A}J$$

to be strictly monotone, bounded and coercive allows to $\mathcal{S}$ to be such that

$$\langle \mathcal{S}'u, u\rangle < 0 \quad \text{for some } u \in \mathcal{V}.$$

Step 1. Suppose first $\varphi = \psi = \eta = 0$. Write the equation $(\mathcal{S}v)' + \mathcal{N}v + \mathcal{A}Jv = f$ as follows:

$$\left[\mathcal{S}v' + \frac{1}{2}\mathcal{S}'v\right] + \left[\frac{1}{2}\mathcal{S}'v + \mathcal{N}v + \mathcal{A}Jv\right] = f.$$

By Proposition 3.3 the operator $\mathcal{L}: \mathcal{W}^0 \to \mathcal{V}'$, $\mathcal{L}v = \mathcal{S}v' + \frac{1}{2}\mathcal{S}'v$, is maximal monotone; the operator $\mathcal{B}: \mathcal{W}^0 \to \mathcal{V}'$, $\mathcal{B}v = \frac{1}{2}\mathcal{S}'v + \mathcal{N}v + \mathcal{A}Jv$ is monotone, bounded and coercive by assumption. Then we can use Theorem 2.5 to conclude that

$$\begin{cases} (\mathcal{S}v)' + \mathcal{N}v + \mathcal{A}Jv = f, \\ P_+(0)v(0) = 0, \\ P_-(T)v(T) = 0 \end{cases}$$

has one solution. If $\mathcal{B}$ is strictly monotone the solution is unique.

Step 2. Suppose $\mathcal{B}$ is strictly monotone and consider first $\Phi, \Psi \in V$ with $P_+(0)\Phi = \varphi$, $P_-(T)\Psi = \psi$, $P_+(0)\Psi = 0$, $P_-(T)\Phi = 0$ (and still $\eta = 0$). This is possible because of (26). We first define $\vartheta = \Phi + \Psi$ and solve

$$\begin{cases} (\mathcal{S}w)' + \mathcal{N}w + \mathcal{A}Jw = f - \mathcal{S}'\vartheta - \mathcal{N}\vartheta - \mathcal{A}J\vartheta, \\ P_+(0)w(0) = 0, \\ P_-(T)w(T) = 0. \end{cases}$$

We can solve this problem by Step 1. Then, denoting by w a solution, $v = w + \vartheta$ will be a solution of the first order equation in (28) since, thanks to (26), $P_+(0)\vartheta = P_+(0)\Phi$ and $P_-(T)\vartheta = P_-(T)\Psi$.

Step 3. Suppose now $\mathcal{N}$ is continuous, (24) and (27) to be true and consider $\varphi \in \tilde{H}_+(0)$ and $\psi \in \tilde{H}_-(T)$. Consider two sequences $(\Phi_n)_n \subset V_+(0)$ and $(\Psi_n)_n \subset V_-(T)$ such that (thanks to (27))

$$\varphi_n := P_+(0)\Phi_n \to \varphi \quad \text{in } \tilde{H}_+(0), \qquad \psi_n := P_-(T)\Psi_n \to \psi \quad \text{in } \tilde{H}_-(T),$$

$f \in \mathcal{V}'$, $\eta = 0$ and the problems:

$$\begin{cases} (\mathcal{S}u')' + \mathcal{N}u' + \mathcal{A}u = f, \\ P_+(0)u'(0) = \varphi_n, \\ P_-(T)u'(T) = \psi_n, \\ u(0) = 0. \end{cases}$$

Denote by v_n the solutions (by Step 2) of the problems:

$$\begin{cases} (\mathcal{S}v)' + \mathcal{N}v + \mathcal{A}Jv = f, \\ P_+(0)v(0) = \varphi_n, \\ P_-(T)v(T) = \psi_n. \end{cases}$$

Consider the $v_n - v_k$: since the problems above are linear the function $v_n - v_k$ solves the problem:

$$\begin{cases} (\mathcal{S}v)' + \mathcal{N}v + \mathcal{A}Jv = 0, \\ P_+(0)v(0) = \varphi_n - \varphi_k, \\ P_-(T)v(T) = \psi_n - \psi_k. \end{cases}$$

Multiplying the equation by $v_n - v_k$ we obtain

$$\begin{aligned} &\left\langle \left[\frac{1}{2}\mathcal{S}' + \mathcal{N} + \mathcal{A}J\right](v_n - v_k), v_n - v_k \right\rangle_{\mathcal{V}'\times\mathcal{V}} \\ &\quad = -\left\langle \mathcal{S}(v_n - v_k)' + \frac{1}{2}\mathcal{S}'(v_n - v_k), v_n - v_k \right\rangle_{\mathcal{V}'\times\mathcal{V}} \\ &\quad = \frac{1}{2}\Big[\big(S(0)(v_n - v_k)(0), (v_n - v_k)(0)\big)_H - \big(S(T)(v_n - v_k)(T), (v_n - v_k)(T)\big)_H\Big] \\ &\quad \leqslant \frac{1}{2}\Big[\big(S_+(0)(v_n - v_k)(0), (v_n - v_k)(0)\big)_H + \big(S_-(T)(v_n - v_k)(T), (v_n - v_k)(T)\big)_H\Big]. \end{aligned}$$

By assumption this term is going to zero for $n, k \to +\infty$. In particular, by assumption, $\gamma + \Gamma/2 > 0$ and by (25) we have

$$\begin{aligned} (\gamma + \Gamma/2)\|v_n - v_k\|_{\mathcal{V}}^2 &\leqslant \left\langle \left[\frac{1}{2}\mathcal{S}' + \mathcal{N}\right](v_n - v_k), v_n - v_k \right\rangle_{\mathcal{V}'\times\mathcal{V}} \\ &\leqslant \left\langle \left[\frac{1}{2}\mathcal{S}' + \mathcal{N} + \mathcal{A}J\right](v_n - v_k), v_n - v_k \right\rangle_{\mathcal{V}'\times\mathcal{V}}. \end{aligned}$$

Then $\{v_n\}_n$ is a Cauchy sequence in $\mathcal{V}$ and consequently

$$v_n \to v \quad \text{and} \quad Jv_n \to Jv \quad \text{in } \mathcal{V}.$$

By Proposition 3.2 we get

$$\begin{aligned} &-(\mathcal{S}v_n, \Phi')_{\mathcal{H}} + \langle \mathcal{N}v_n, \Phi\rangle_{\mathcal{V}'\times\mathcal{V}} + \langle \mathcal{A}Jv_n, \Phi\rangle_{\mathcal{V}'\times\mathcal{V}} \\ &\quad = \langle f, \Phi\rangle_{\mathcal{V}'\times\mathcal{V}} + \big(S_+(0)\varphi_n, \Phi(0)\big)_H - \big(S_-(T)\psi_n, \Phi(T)\big)_H \end{aligned}$$

for every $\Phi \in \{v \in \mathcal{V} \mid v' \in \mathcal{H},\ P_-(0)v = 0,\ P_+(T)v = 0\}$. Taking the limit and by the continuity of $\mathcal{S}$, $\mathcal{N}$ and $\mathcal{A}J$ we obtain a limit equation satisfied by the function v with $\varphi \in \tilde{H}_+(0)$ and $\psi \in \tilde{H}_-(T)$.

Step 4. Finally, if $\eta \in V$ is not null, we consider u the solution of the problem:

$$\begin{cases} (\mathcal{S}u')' + \mathcal{N}u' + \mathcal{A}u = f - \mathcal{A}\eta, \\ P_+(0)u'(0) = \varphi, \\ P_-(T)u'(T) = \psi, \\ u(0) = 0 \end{cases}$$

with $f \in \mathcal{V}'$, $\varphi \in \tilde{H}_+(0)$, $\psi \in \tilde{H}_-(T)$, and then consider $\eta + u(t)$, which solves (16).

To conclude and prove the estimate observe that, by Theorem 3.8 in [8], we have that

$$\left\|(\mathcal{S}u')'\right\|_{\mathcal{V}'} + \|u'\|_{\mathcal{V}} \leqslant c\left(\|f\|_{\mathcal{V}'} + \left\|S_-^{1/2}(T)\psi\right\|_{H_-(T)} + \left\|S_+^{1/2}(0)\varphi\right\|_{H_+(0)}\right)$$

where $c = c(C_5, \Gamma/2 + \gamma)$. Finally we conclude by

$$\left\|u(t)\right\|_V = \left\|\eta + \int_0^t u'(\sigma)\,d\sigma\right\|_V \leqslant \|\eta\|_V + T^{1/2}\|u'\|_{\mathcal{V}}. \qquad \square$$

Here we present a concrete example, just to explain the assumptions about the operators $\mathcal{S}$, $\mathcal{N}$, $\mathcal{A}$, for further examples of possible choices of the function s in (29) we refer to [8].

Consider $T > 0$, Ω open bounded subset of $\mathbf{R}^n$ with Lipschitzian boundary and a function

$$s : \Omega \times [0, T] \to \mathbf{R}.$$

By s_+ and s_- we denote respectively the non-negative and non-positive part of s. Consider the following problem:

$$\begin{cases} (su_t)_t - \operatorname{div}(\alpha(|Du_t|^2)Du_t) + ru_t - \operatorname{div}(a \cdot Du) = f(x,t) \quad \text{on } \Omega \times (0,T), \\ u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \\ \int_\Omega (u_t(x,0) - \varphi(x))^2 s_+(x,0)\,dx = 0, \\ \int_\Omega (u_t(x,T) - \psi(x))^2 s_-(x,T)\,dx = 0, \\ u(x,0) = \eta(x), \quad x \in \Omega. \end{cases} \tag{29}$$

The initial/final conditions $P_+(0)u'(0) = \varphi$ and $P_-(T)u'(T) = \psi$ in (16) are to be understood as the integral conditions involving φ and ψ in (29).

Notice that if $s \equiv 1$, $r \equiv 0$ and $a_{ij} = \delta_{ij}$ this equation is the wave equation with a dissipative term and with initial condition φ for u_t on the whole Ω (the condition involving ψ disappears).

This problem can be thought of as a model problem for (16): consider $V = H_0^1(\Omega)$, $H = L^2(\Omega)$, $\mathcal{V} = L^2(0,T; H_0^1(\Omega))$, $\mathcal{H} = L^2(0,T; L^2(\Omega)) = L^2(\Omega \times (0,T))$ and

$$S(t) : H \to H \quad \text{defined by} \quad S(t)u = s(x,t)u(x)$$

and then

$$\mathcal{S} : \mathcal{H} \to \mathcal{H} \quad \text{defined by} \quad \mathcal{S}u(t)(x) = s(x,t)u(x,t). \tag{30}$$

In order that (9) is satisfied, the function s should be such that

$$t \mapsto \int_\Omega u(x)v(x)s(x,t)\,dx \quad \text{is absolutely continuous,}$$

and for this reason we can consider the integral conditions involving φ and ψ in (29), and

$$\Gamma \int_\Omega \left|Du(x)\right|^2 dx \leqslant \frac{d}{dt} \int_\Omega u^2(x) s(x,t)\, dx \leqslant C_2 \int_\Omega \left|Du(x)\right|^2 dx$$

with $\Gamma \in \mathbf{R}$, $C_2 \geqslant 0$ (for more details we refer to [8]). The data will satisfy: $f \in L^2(0,T; H^{-1}(\Omega))$, $\varphi \in L^2(\Omega_+(0), s_+(\cdot,0))$, $\psi \in L^2(\Omega_-(T), s_-(\cdot,T))$, $\eta \in H_0^1(\Omega)$ (for the choice of φ and ψ in such spaces we refer to [8]). The operator $\mathcal{N}$ is defined by

$$\mathcal{N}: \mathcal{V} \to \mathcal{V}', \quad \mathcal{N} w(t)(x) = -\operatorname{div}\big(\alpha\big(\left|Dw(x,t)\right|^2\big) Dw(x,t)\big)$$

with α satisfying

$$\alpha: [0,+\infty) \to [0,C], \quad \alpha\big(\tau^2\big)\tau - \alpha\big(\sigma^2\big)\sigma \geqslant \gamma(\tau - \sigma)$$

where C is a positive constant, γ is the constant appearing in (24) and $0 \leqslant \sigma < \tau$. The function $r = r(x)$ belongs to $L^\infty(\Omega)$ and finally

$$\mathcal{A}: \mathcal{V} \to \mathcal{V}', \quad \mathcal{A} w(t)(x) = -\operatorname{div}\big(a(x,t) \cdot Du(x,t)\big)$$

with $a = [a_{ij}(x,t)]_{i,j=1}^n$ satisfying (so that assumptions of Proposition 2.3 is satisfied)

$$a_{ij} = a_{ji}, \quad \int_0^T \int_\Omega \sum_{i,j=1}^n (a_{ij})_t D_i u D_j u \, dx\, dt \leqslant 0.$$

The last condition is satisfied, for instance, if $a_{ij} = a_{ij}(x)$. The simpler situation could be the following: if

$$\alpha(\tau) \equiv \gamma \in \mathbf{R}, \qquad r \equiv 0, \qquad a_{ij} = \delta_{ij}$$

problem (29) becomes

$$\begin{cases} (su_t)_t - \gamma \Delta u_t - \Delta u = f(x,t) & \text{on } \Omega \times (0,T), \\ u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \\ \int_\Omega (u_t(x,0) - \varphi(x))^2 s_+(x,0)\, dx = 0, \\ \int_\Omega (u_t(x,T) - \psi(x))^2 s_-(x,T)\, dx = 0, \\ u(x,0) = \eta(x), \quad x \in \Omega. \end{cases} \tag{31}$$

In this situation we may replace the last initial condition by $u(x,T) = \eta(x)$ in Ω, and it has a unique solution (see Remark 3.6).

We conclude by observing that if the constant Γ in (9)(iv) is strictly positive we can admit also the function α not to be non-negative. For instance, consider for simplicity example (31), we can admit also $\gamma \leqslant 0$ and consider in particular the equation

$$(su_t)_t - \Delta u = f$$

(where indeed, if s is regular enough, we have $su_{tt} + s_t u_t - \Delta u = f$, so a term involving u_t appears in any case).

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